Universitá degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di AM210

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1. Notiamo che

$$\frac{\partial}{\partial x}\left[\left(x+\frac{2}{t}\right)\frac{e^{-tx}}{t^2}\right] = -\frac{e^{-tx}}{t}\left(x+\frac{1}{t}\right)$$

che é equidominata da $g(t)=\frac{1}{t^2}$ essendo:

$$\lim_{t\to+\infty}\frac{-\frac{e^{-tx}}{t}\left(x+\frac{1}{t}\right)}{\frac{1}{t^2}} = -\lim_{t\to+\infty} te^{-xt}\left(x+\frac{1}{t}\right) = 0\ .$$

Pertanto

$$f(x)=\int_1^{+\infty}\left(x+\frac{2}{t}\right)\frac{e^{-tx}}{t^2}\ dt \Longrightarrow f'(x)=\int_1^{+\infty}-\frac{e^{-tx}}{t}\left(x+\frac{1}{t}\right)\ dt\ .$$

Ma

$$\frac{\partial}{\partial x}\left[-\frac{e^{-tx}}{t}\left(x+\frac{1}{t}\right)\right] = xe^{-tx}$$

che é dominata ancora da $g(t)$. Quindi

$$f''(x)=\int_1^{+\infty} xe^{-tx}\ dt = \left[-e^{-tx}\right]_1^{+\infty} = e^{-x}.$$

Pertanto

$$f(x)=\int_1^{+\infty}\left(x+\frac{2}{t}\right)\frac{e^{-tx}}{t^2}\ dt = e^{-x}.$$

NB. Essendo $f''(x)=e^{-x}$, scrivere $f(x)=e^{-x}$ non é del tutto corretto. Difatti

$$f''(x)=e^{-x}\Longrightarrow f'(x)=-e^{-x}+c \Longrightarrow f(x)=e^{-x}+cx+k\ .$$

Tuttavia, essendo $f(0)=\displaystyle\int_1^{+\infty}\frac{2}{t^3}\ dt = \left[-\frac{1}{t^2}\right]_1^{+\infty} = 1$ e
$f'(0)=\displaystyle\int_1^{+\infty}-\frac{1}{t^2}\ dt = \left[\frac{1}{t}\right]_1^{+\infty} = -1$ abbiamo che

$$\begin{cases} f(0)=1+k=1 \\ f'(0)=-1+c=-1 \end{cases} \Longrightarrow c=k=0 \Longrightarrow f(x)=e^{-x}\ .$$

2. Procediamo come nella risoluzione dell'analogo esercizio del precedente tutorato:

   (a) Essendo $f(x) = 2x,\ x \in (-1, 1]$, una funzione dispari, abbiamo che $a_n = 0\ \forall\ n$.

$$b_n = \frac{4}{\pi}\int_0^1 x\sin(nx)\ dx = \frac{4}{\pi}\left\{\left[-\frac{x\cos(nx)}{n}\right]_0^1 + \int_0^1 \frac{\cos(nx)}{n}\ dx\right\} \Longrightarrow$$

$$\Longrightarrow b_n = \frac{4}{\pi}\left(-\frac{\cos(n)}{n} + \frac{\sin(n)}{n^2}\right).$$

   Pertanto

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{\sin(nx)}{n^2}\left(\sin(n) - n\cos(n)\right)\ .$$

   (b) Come sopra si ha che $g(x)$ é una funzione dispari, quindi $a_n = 0\ \forall\ n$.

$$b_n = \frac{2}{\pi}\int_0^\pi (\pi x - |x|x)\sin(nx)\ dx = 2\int_0^\pi x\sin(nx)\ dx +$$

$$-\frac{2}{\pi}\int_0^\pi x^2\sin(nx)\ dx = 2\left\{\left[-\frac{x\cos(nx)}{n}\right]_0^\pi + \int_0^\pi \frac{\cos(nx)}{n}\ dx\right\} +$$

$$-\frac{2}{\pi}\left\{\left[-\frac{x^2\cos(nx)}{n}\right]_0^\pi + 2\int_0^\pi \frac{x\cos(nx)}{n}\ dx\right\} = \frac{2\pi(-1)^{n+1}}{n} +$$

$$-\frac{2}{\pi}\left\{\frac{\pi^2(-1)^{n+1}}{n} + 2\left\{\left[\frac{x\sin(nx)}{n^2}\right]_0^\pi - \int_0^\pi \frac{\sin(nx)}{n^2}\ dx\right\}\right\} =$$

$$= \frac{4}{\pi n^2}\int_0^\pi \sin(nx)\ dx = -\frac{4}{\pi n^3}\left[\cos(nx)\right]_0^\pi = -\frac{4((-1)^n - 1)}{\pi n^3} \Longrightarrow$$

$$\Longrightarrow b_n = \begin{cases} \frac{8}{\pi(2k+1)^3} & \text{se}\ \ n = 2k+1 \\ 0 & \text{se}\ \ n = 2k \end{cases}$$

   Pertanto

$$\pi x - |x|x = \frac{8}{\pi}\sum_{n=0}^{+\infty}\frac{\sin[(2n+1)x]}{(2n+1)^3}\ .$$

   Sostituendo $x = \frac{\pi}{2}$ nell'uguaglianza ottenut, abbiamo che

$$\frac{\pi^2}{2} - \frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3} \Longrightarrow \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

   (c)

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^0 (2-x)\ dx + \int_0^\pi 2\ dx\right) = \frac{2}{\pi}\left(\int_{-\pi}^\pi\ dx - \int_{-\pi}^0 \frac{x}{2}\ dx\right) =$$

$$= \frac{2}{\pi}\left(2\pi - \left[\frac{x^2}{4}\right]_{-\pi}^{0}\right) = 4 + \frac{\pi}{2} \ ;$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (2-x)\cos(nx)\ dx + \int_{0}^{\pi} 2\cos(nx)\ dx\right) = -\frac{1}{\pi}\int_{-\pi}^{0} x\cos(nx)\ dx =$$

$$= \frac{1}{n\pi}\int_{-\pi}^{0} \sin(nx)\ dx = -\frac{1}{n^2\pi}\left[\cos(nx)\right]_{-\pi}^{0} = -\frac{1}{n^2\pi} + \frac{(-1)^n}{n^2\pi} \Longrightarrow$$

$$\Longrightarrow \begin{cases} a_{2k} = 0 \\ a_{2k+1} = -\frac{2}{(2k+1)^2\pi} \end{cases} \quad ;$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (2-x)\sin(nx)\ dx + \int_{0}^{\pi} 2\sin(nx)\ dx\right) =$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} 2\sin(nx)\ dx - \int_{-\pi}^{0} x\sin(nx)\ dx\right) = \frac{1}{\pi}\left[\frac{x\cos(nx)}{n}\right]_{-\pi}^{0} = \frac{(-1)^n}{n} \ .$$

Quindi

$$h(x) = 2 + \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{+\infty}\frac{\cos[(2n+1)x]}{(2n+1)^2} + \sum_{n=1}^{+\infty}\frac{(-1)^n\sin(nx)}{n} \ .$$

3. Essendo

$$e^x = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+1}\left(\cos(nx) - n\sin(nx)\right)$$

per il precedente tutorato, abbiamo che

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\ dx = \frac{2\sinh^2(\pi)}{\pi^2} + \sum_{n=1}^{+\infty}\frac{4\sinh^2(\pi)}{\pi^2(n^2+1)^2} + \frac{4\sinh^2(\pi)n^2}{\pi^2(n^2+1)^2} =$$

$$= \frac{2\sinh^2(\pi)}{\pi^2} + \frac{4\sinh^2(\pi)}{\pi^2}\ \sum_{n=1}^{+\infty}\frac{1}{n^2+1}$$

Ma

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2x}\ dx = \frac{1}{\pi}\left[\frac{e^{2x}}{2}\right]_{-\pi}^{\pi} = \frac{\sinh(2\pi)}{\pi} \Longrightarrow$$

$$\Longrightarrow \sum_{n=1}^{+\infty}\frac{1}{n^2+1} = \left(\frac{\sinh(2\pi)}{\pi} - \frac{2\sinh^2(\pi)}{\pi^2}\right)\frac{\pi^2}{4\sinh^2(\pi)} =$$

$$= \frac{2\pi\sinh(\pi)\cosh(\pi)}{4\sinh^2(\pi)} - \frac{1}{2} = \frac{1}{2}(\pi\coth(\pi) - 1) \ ;$$

Essendo

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

abbiamo che

$$\frac{2}{\pi}\int_0^\pi x^4\ dx = \frac{2}{9}\pi^4 + \sum_{n=1}^{+\infty}\frac{16}{n^4} \Longrightarrow \sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{1}{5}-\frac{1}{9}\right) = \frac{\pi^4}{90}$$

Ma

$$\sum_{n=1}^{+\infty}\frac{1}{n^4} = \sum_{k=1}^{+\infty}\frac{1}{(2k)^4} + \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^4} \Longrightarrow \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{90} - \frac{\pi^4}{90}\frac{1}{16} = \frac{\pi^4}{96}\ ;$$

Essendo $f_3(x)$ dispari, abbiamo che $a_n = 0\ \forall\ n$, mentre

$$b_n = \frac{2}{\pi}\int_0^\pi x\cos(x)\sin(nx)\ dx = \frac{1}{\pi}\left(\int_0^\pi x\sin[(n+1)x]\ dx + \int_0^\pi x\sin[(n-1)x]\ dx\right) =$$

$$= -\frac{1}{\pi}\left\{\left[\frac{x\cos[(n+1)x]}{n+1} + \frac{x\cos[(n-1)x]}{n-1}\right]_0^\pi\right\} = \frac{2n(-1)^n}{n^2-1},\quad n \neq 1,$$

mentre

$$b_1 = \frac{1}{\pi}\int_0^\pi x\sin(2x)\ dx = -\frac{1}{\pi}\int_0^\pi \sin^2(x)\ dx = -\frac{1}{2}\ ,$$

Pertanto

$$\frac{2}{\pi}\int_0^\pi x^2\cos^2(x)\ dx = \frac{1}{4} + 4\sum_{n=2}^{+\infty}\frac{n^2}{(n^2-1)^2}\ .$$

Ma

$$\int_0^\pi x^2\cos^2(x)\ dx = \left[x^2\frac{(x+\sin(x)\cos(x))}{2}\right]_0^\pi - \int_0^\pi x^2\ dx - \frac{1}{2}\int_0^\pi x\sin(2x)\ dx =$$

$$= \frac{\pi^3}{2} - \frac{\pi^3}{3} + \frac{\pi}{4} = \frac{\pi^3}{6} + \frac{\pi}{4} \Longrightarrow$$

$$\Longrightarrow \sum_{n=2}^{+\infty}\frac{n^2}{(n^2-1)^2} = \frac{1}{4}\left(\frac{2}{\pi}\left(\frac{\pi^3}{6}+\frac{\pi}{4}\right) - \frac{1}{4}\right) = \frac{1}{4}\left(\frac{\pi^2}{3}+\frac{1}{4}\right) = \frac{4\pi^2+3}{48}\ .$$

4. Lo sviluppo di $\cos(\frac{x}{2})$ esteso per periodicitá al di fuori di $[0, 2\pi)$, fá sí che la funzione risulti essere dispara in $[-\pi, \pi]$. Pertanto $a_n = 0\ \forall\ n$, mentre

$$b_n = \frac{2}{\pi}\int_0^\pi \cos\left(\frac{x}{2}\right)\sin(nx)\ dx = \frac{1}{\pi}\left(\int_0^\pi \sin\left(\left[\frac{2n+1}{2}\right]x\right)\ dx + \int_0^\pi \sin\left(\left[\frac{2n-1}{2}\right]x\right)\ dx\right) =$$

$$= -\frac{2}{\pi}\left(\left[\frac{\cos\left(\left[\frac{2n+1}{2}\right]x\right)}{2n+1} + \frac{\cos\left(\left[\frac{2n-1}{2}\right]x\right)}{2n-1}\right]_0^\pi\right) = \frac{2}{\pi}\left(\frac{1}{2n+1}+\frac{1}{2n-1}\right) = \frac{8n}{\pi(4n^2-1)}$$

Quindi

$$\cos\left(\frac{x}{2}\right) = \frac{8}{\pi}\sum_{n=1}^{+\infty}\frac{n}{4n^2-1}\sin(nx)\ .$$

Per l'identitá di Parseval abbiamo che

$$\frac{2}{\pi}\int_0^\pi \cos^2\left(\frac{x}{2}\right)\ dx = \frac{64}{\pi^2}\sum_{n=1}^{+\infty}\frac{n^2}{(4n^2-1)^2}$$

Essendo

$$\frac{2}{\pi}\int_0^{\pi} \cos^2\left(\frac{x}{2}\right)\ dx =_{\left(\frac{x}{2}=t\right)} \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos^2(t)\ dt = 1 \Longrightarrow$$

$$\Longrightarrow \sum_{n=1}^{+\infty} \frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}\ .$$

5. Notiamo che

$$\frac{\partial}{\partial t}\tan(xt) = \frac{x}{\cos^2(xt)}\ .$$

Pertanto, definita

$$f'(t) = \int_0^{\frac{\pi}{4}} \frac{x}{\cos^2(xt)}\ dx = \int_0^{\frac{\pi}{4}} \frac{\partial}{\partial t}\tan(xt)\ dx$$

si ha che

$$f(t) = \int_0^{\frac{\pi}{4}} \tan(xt)\ dx = -\frac{1}{t}\int_0^{\frac{\pi}{4}} \frac{-t\sin(xt)}{\cos(xt)}\ dx = \left[-\frac{1}{t}\log(\cos(xt))\right]_0^{\frac{\pi}{4}} =$$

$$= -\frac{1}{t}\log\left(\cos\left(\frac{\pi t}{4}\right)\right)\ ;$$

Dunque

$$f'(t) = \frac{1}{t^2}\log\left(\cos\left(\frac{\pi t}{4}\right)\right) + \frac{\pi}{4t}\tan\left(\frac{\pi t}{4}\right)$$

e perció

$$f'(1) = \int_0^{\frac{\pi}{4}} \frac{x}{\cos^2(x)}\ dx = \log\left(\frac{\sqrt{2}}{2}\right) + \frac{\pi}{4} = -\frac{1}{2}\log(2) + \frac{\pi}{4} = \frac{\pi - \log(4)}{4}\ .$$